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A method developed by the present authors in a previous paper [Phys. Rev. E **57**, 2594 (1998)] leads to the introduction of the equilibrium size of the Kadanoff blocks as a useful tool to approach the critical properties of the ϕ^4 model. The present paper aims to elucidate the role of the equilibrium size of the Kadanoff blocks in the loop-expansion technique currently used in the field-theoretic renormalization. While the standard results are readily obtained, aspects emerge that help clarify the true nature of the smallness parameter in the loop-expansion technique. [S1063-651X(98)00911-8]

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I. INTRODUCTION

In a recent paper [1] we introduced an unconventional renormalizative approach to the ϕ^4 model, based on the *residual* free energy. It is shown therein that if θ is the relevant field, there is a special value $s^*(\theta)$ of the scaling parameter s having the same *critical* properties of divergence as the correlation length ξ.

Instead of calculating the second moment of the pair correlation function (the usual definition of ξ), $s^*(\theta)$ is obtained by minimizing the residual free energy $f_{\rm res}(s,\theta)$, which follows from Wilson's renormalization procedure reducing the original number of degrees of freedom by a factor s^d (in d dimension). Since the residual free energy $f_{\rm res}(s)$ can be interpreted as the ''formation energy'' of Kadanoff blocks of linear size s, it turns out that s^* is the equilibrium size of the Kadanoff blocks, provided they are considered as *canonic* systems exchanging *heat* with one another. This argument substantiates on a physical ground the relationship, obtained in [1] as a formal result, between the correlation length ξ and s^*.

The equilibrium size s^* of the Kadanoff blocks is obviously derived from Wilson renormalization group theory (WRGT) [2]. In the present paper we will show that s^* also enters the field-theoretic approach to renormalization, in a significant way, since $1/s^*$ actually plays the role of the additional parameter to be introduced for *massless* field renormalization. The textbook of Amit [3] will be taken as a reference point in what follows.

The main advantage of the present method is that the correlation length (that is, s^*) enters the calculations with its own physical meaning, while in the standard field-theoretic renormalization (FTR), the scaling parameter s is introduced *ad hoc*, and somehow arbitrarily, in order to remedy the infrared divergences arising in a massless theory. Since the correlation length is a physical quantity, we have no arbitrariness at all. In particular, we will show that introducing the correlation length actually produces noninfrared divergences, which are inherent to the cumulant expansion rather than to the vanishing of the mass. However, the loop-expansion technique suggests a way to shift those divergences to next-order terms, by redefining a ''dressed'' relevant field (renormalized mass) order by order. As a consequence, the present method makes it possible to find both the critical point and the critical exponent ν by expressing s^* as a function of the dressed field and then by studying the limiting case $s^*\to\infty$. It will be shown that the failure of the loop expansion is characterized just by the impossibility for s^* to diverge at any finite value of the relevant field.

In order to implement the connection between s^* and FTR, we first show that the resummation of the diagrams in Fig. 1 is equivalent to the one-loop approximation in FTR (see [3], Chap. 6). In Sec. II we also show that if r_0 is the relevant field and $r_0=0$ is the *Gaussian* critical value, then the resummation of the diagrams in Fig. 1 yields a new critical point, shifted below by the quantity

$$r_c=\frac{12d}{d-2}u_0 \quad (d=3,4,5,\dots). \tag{1}$$

This coincides with the results of the one-loop approximation. In addition, the values in Eq. (1) turn out to be the lowest-order approximants (in u_0) for the exact critical value $-\theta_c$ (Sec. IV). The new relevant field is thereby conveniently defined as

$$\theta\equiv r_0+\theta_c=r_0+r_c+O(u_0^{1+a}), \tag{2}$$

with a a positive exponent. As far as the critical exponents are concerned, we find again the standard results of the one-loop approximation, that is, the Ginzburg criterion (Sec. III). In addition, we find that the one-loop approximation maps the one-component model (discrete symmetry) onto the spherical model (continuous symmetry) in any dimension. To the authors' knowledge, this point has never been

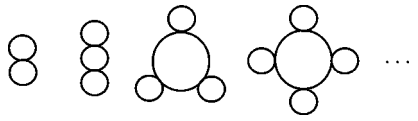

FIG. 1. Vacuum bubble diagrams containing, order by order, the maximum number of tadpole subdiagrams.

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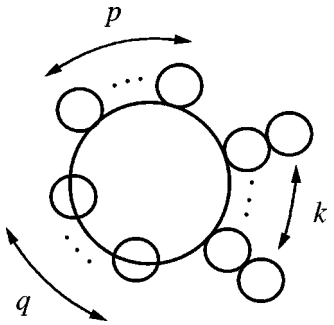

FIG. 2. Ring diagram dressed with all the possible combinations of p tadpole, k double-tadpole, and q setting-sun subdiagrams $[p,k,q \in \mathrm{N}$, but $(p,q,k) \neq (0,0,0)]$.

stressed before, though some results pointing to the same conclusion have already been reported in the literature (see, for example, [3]). The reason is probably that the standard method of FTR does not provide the complete one-loop solution in $d=2$, while our method does. We actually find that in $d=2$ the one-loop approximation yields the same results as the Mermin-Wagner theorem [4] for the spherical model, that is, the absence of ordered phases at any finite temperature.

In Sec. V we accomplish our program by identifying those diagrams whose resummation is equivalent to the two-loop approximation in FTR (see Fig. 2). It is found that in this case no ordered phase can be found at finite temperature, even in $d=3$. This makes it clear that the loop expansion does not improve our knowledge of the critical phenomenon with increasing number of loops. The reason is that the true smallness parameter δ_n of the loop expansion does contain both u_0 *and* s^*. This is shown in Sec. V, where the general expression of δ_n at the nth loop is given. It is seen that in $d \leqslant 3$ the divergence of s^* leads δ_n to diverge as well, for $n>1$. This is the crucial aspect that makes the loop expansion unsuitable in $d \leqslant 3$, unless other perturbative techniques (such as the ϵ expansion) are implemented.

II. DRESSING THE RELEVANT FIELD AT ONE-LOOP ORDER

For the sake of simplicity, we refer to an *abstract* one-component ϕ^4 model, in zero external field, characterized by the two independent parameters r_0, u_0, with the Hamiltonian

$$\beta H = \frac{1}{N} \sum_{q \in B} (r_0 + q^2) |\phi_{\vec{q}}|^2 + \frac{u_0}{N^3} \sum_{q_i \in B} \phi_{\vec{q}_1} \phi_{\vec{q}_2} \phi_{\vec{q}_3} \phi_{\vec{q}_4} \times \delta(\vec{q}_1 + \vec{q}_2 + \vec{q}_3 + \vec{q}_4) \tag{3}$$

in (dimensionless) momentum space. B is the (dimensionless) Brillouin zone and N is the total number of degrees of freedom. A word of caution is in order about the procedure adopted to eliminate the factor $\beta = 1/k_B T$ in Eq. (3) and about the presence of N. The latter feature is necessary because the shrinking of the number of degrees of freedom is an important aspect to be accounted for in the calculation of the residual free energy $f_{\mathrm{res}}(s,r_0)$ [1]. In this case, it is convenient to perform a linear change in the integral measure, leading to the definition of the partition function Z,

$$Z = (\pi N \gamma)^{-N/2} \int d\{\phi_{\vec{q}}\} e^{-\beta H}, \tag{4}$$

with $\gamma \propto \beta = 1/k_B T$ (see details in [1]). In what follows we take γ as a *fixed* quantity, but let r_0 change arbitrarily, since we are interested in the critical value of the relevant field. The critical value of the *temperature* is, of course, another matter of affairs, which should keep track of the dependence of γ on the temperature and of the mapping between the specific *physical* model and the abstract model Eqs. (3) and (4).

It should be stressed that, from the very definition (3), the Gaussian critical value corresponds to $r_0 = 0$ and $\theta = \theta_c$ if the relevant field θ is defined by Eq. (2). On performing a single renormalization procedure on Eq. (4) and expanding in Gaussian cumulants, the residual free energy in d dimensions takes the form [1]

$$\beta f_{\mathrm{res}}(s,r_0) = \beta f_{\mathrm{res}}^{(0)}(s,r_0) + \sum_{n=1}^{\infty} \frac{u_0^n}{n!} \mathcal{G}_n(s,r_0), \tag{5}$$

where

$$\beta f_{\mathrm{res}}^{(0)}(s,r_0) = \frac{d}{2} \underbrace{\int_{1/s}^{1} \mathrm{d}x \; x^{d-1} \ln\left(r_0 + x^2\right)}_{\Phi_d(s,r_0)} + \frac{1}{2}\left(1 - \frac{1}{s^d}\right) \ln \gamma - \frac{\ln s}{s^d}, \tag{6}$$

is the Gaussian residual free energy. In [1] the expression of $\mathcal{G}_1$ (corresponding to the first diagram in Fig. 1) has been explicitly calculated, with the result

$$\mathcal{G}_1(s,r_0) = 3d^2 [F_d(s,r_0)]^2, \quad F_d(s,r_0) = \frac{\partial \Phi_d(s,r_0)}{\partial r_0}. \tag{7}$$

The next terms of the expansion (5) contain contributions from an increasing number of diagrams. However, the diagrams indicated in Fig. 1 have the following important property: At $r_0 = 0$, they yield, order by order, the *maximum* power s^{2n-d} of the scaling parameter. They are thereby dominant when r_0 is small and the calculated value of s^* is large. This is the reason

why we keep them under special control, when we study how an arbitrarily small u_0 removes the Gaussian singularity at $r_0=0$, possibly shifting it below. The details of the resummation can be found in Ref. [5] and the resulting expression is

$$\beta f_{\rm res}(s,r_0)=\underbrace{\beta f_{\rm res}^{(0)}(s,r_0+12du_0F_d(s,r_0))-3d^2u_0[F_d(s,r_0)]^2}_{\beta f_{\rm res}^{(1)}}+(\text{Contributions from Other Diagrams}), \tag{8}$$

showing that the residual free energy of the ϕ^4 model actually contains a *shifted* Gaussian term resulting from the diagrams in Fig. 1. The second argument of the logarithmic part of Φ_d [Eq. (6)] is $r_0+12du_0F_d(s,r_0)$, i.e., the *direct* correlation function (otherwise denoted as the *two-point vertex function* in FTR), at first order in u_0. It is important to stress that the function $\Phi_d(s,r_0)$, from which the diagrams in Fig. 1 originate, is *singular*, if $r_0<0$, at the point $s=1/\sqrt{|r_0|}$. It is at this stage that our method differs from the standard FTR. We know that s will be replaced by s^* [obtained on minimizing the free energy (8)] and that s^* behaves like the correlation length. Hence s^* will *unavoidably* cross the singularity point $1/\sqrt{|r_0|}$ if the critical point corresponds to a negative value of r_0. Note that the divergence of $F_d(s^*,r_0)$ [Eq. (7)] is not related to the vanishing of the ''bare mass'' ($r_0=0$), but to the cumulant expansion on a Gaussian distribution, when some eigenvalues of the quadratic form become negative. Hence we need a formal prescription to extend the calculations in the region $r_0<0$ (negative bare mass), with $s\geqslant 1/\sqrt{|r_0|}$. The prescription we use here follows the same line of reasoning as the one-loop approximation in FTR (see [3]). We define a ''dressed'' relevant field through the implicit equation

$$r_1=r_0+12du_0F_d(s,r_1)\quad (r_1\geqslant 0), \tag{9}$$

which does not contain any singular integral. In addition, the difference r_1-r_0 is formally small to first order in u_0. The procedure we adopt is simply to express r_0 in Eq. (8), in terms of r_1 [Eq. (9)], dropping all higher-order terms in u_0. This yields

$$\beta f_{\rm res}^{(1)}(s,r_0)=\beta f_{\rm res}^{(1)}(s,r_1)+O(u_0^2). \tag{10}$$

At this stage, the strategy to remove the singularities in the integrals is straightforward: Dressing the relevant field (renormalizing the mass) actually shifts the singular terms to the next order, taking advantage of the fact that these terms will be readsorbed into the renormalized quantities in the next-order approximation and so on. This will be explicitly seen at the two-loop level (Sec. V).

Suppose now that we work in a region of parameter space where the contributions to Eq. (8) from the other diagrams is negligible. The value s^* that minimizes the free energy $f_{\rm res}^{(1)}(s,r_1)$ is found through the equation

$$\partial_s f_{\rm res}^{(1)}(s,r_1)+\partial_{r_1}f_{\rm res}^{(1)}(s,r_1)\partial_s r_1=0. \tag{11}$$

From Eqs. (6), (7), and (9) it can be seen that, to first order in u_0, the second term in Eq. (11) is exactly canceled by the non-Gaussian part of the first term, whence Eq. (11) becomes

$$\frac{d}{2}\ln[1+(s^*)^2r_1]=\left(1-\frac{d}{2}\ln\,\gamma\right)\Rightarrow s^*=\frac{c_d}{\sqrt{r_1}};$$

$$c_d\equiv\left(\frac{e^{2/d}}{\gamma}-1\right)^{1/2}, \tag{12}$$

showing that the solution is still Gaussian-like [see [1], Eqs. (14) and (15)], except for replacing the Gaussian field r_0 with the dressed field $r_1(s^*,r_0)$. On the other hand, the dressed field (9) is nothing but the renormalized mass at one-loop order in FTR [see [3], Eq. (6-35)] and $1/s^*$ plays the role of the so-called subtraction point in the field theory (see [3,6]).

III. THE GINZBURG CRITERION REVISITED

The fact that the dressed field r_1 depends on s^* itself [Eq. (9)] is the distinction between the genuine Gaussian problem and the present one-loop approximation. Indeed, we can now make use of Eq. (12) to eliminate r_1 from Eq. (9) in order to get the relationship between s^* and r_0. The resulting equations are

$$\left(\frac{c_4}{s^*}\right)^2=\left[\frac{(r_0+24u_0)c_4^2}{c_4^2+24u_0}\right]\left\{1+\frac{24u_0c_4^2}{c_4^2+24u_0}\times\ln\left[\frac{(s^*)^2+c_4^2}{1+c_4^2}\right]\right\}^{-1}\quad (d=4), \tag{13a}$$

$$\left(\frac{c_3}{s^*}\right)^2=(r_0+36u_0)-\frac{36u_0}{s^*}\left\{1+c_3\left[\arctan(c_3)-\arctan\left(\frac{c_3}{s^*}\right)\right]\right\}\quad (d=3), \tag{13b}$$

$$\left(\frac{c_2}{s^*}\right)^2=r_0+12u_0\ln\left[\frac{(s^*)^2+c_2^2}{1+c_2^2}\right]\quad (d=2). \tag{13c}$$

It is immediately seen that in four and three dimensions s^* diverges at the point $r_0=-r_c$, with r_c given by Eq. (1). Hence, in view of Eq. (2), we set $r_0+r_c=\theta$. In four dimensions, Eq. (13) yields an asymptotic behavior $(s^*)^{-1}\propto\sqrt{\theta/|\ln\theta|}$. The critical exponent is *always* Gaussian, according to the Ginzburg criterion, but with a *logarithmic* correction (that disappears for $d>4$). In three dimensions, Eq. (13b) can be transformed into a second-order equation in $(s^*)^{-1}$ in the limit $(s^*)^{-1}c_3\ll 1$. In this case, there is a crossover from a Gaussian regime $\sqrt{\theta}\gg 36u_0[c_3^{-1}+\arctan(c_3)]$ in which $(s^*)^{-1}\propto\sqrt{\theta}$, to a non-Gaussian re-

gime $\sqrt{\theta} \ll 36u_0[c_3^{-1} + \arctan(c_3)]$ in which $(s^*)^{-1} \propto \theta$ (which means $\nu = 1$). Using the appropriate scaling relations, this last result turns out to be equivalent to the one-loop equation for the susceptibility χ [following Eq. (6-31) in [3]]. It should be noticed that the exponent $\nu = 1$ is *exact* in $d = 3$ for the *spherical* model (recall that, instead, we are dealing with a one-component model).

The two-dimensional case is peculiar in many regards. The standard application of the Ginzburg criterion leads one to the following conclusion: The quantity that should be kept small, for the Gaussian behavior to be recovered *diverges* logarithmically, unless $u_0 = 0$ [see Eq. (6-31) of [3]]. This means that the critical region is, so to speak, divergently large even though u_0 is arbitrarily small. Equation (13c) substantiates the preceding argument in a more quantitative way. Indeed, it turns out that $(s^*)^{-1} \propto \exp(r_0/24u_0)$ in the limit of large s^*, which means $r_c = \infty$ (or, equivalently, the critical point is shifted down to zero temperature). The absence of a *finite* critical point in $d = 2$ is reminiscent of the Mermin-Wagner theorem [4] claiming that a two-dimensional system with *continuous* symmetry cannot exhibit an ordered phase at finite temperature (if the interactions are short ranged). As in the case $d = 3$, the one-loop approximation in $d = 2$ for the one-component model (discrete symmetry) yields the same result as the *exact* theory for the spherical model (continuous symmetry). A general property thus emerges from the present revisitation of the Ginzburg criterion: The one-loop approximation maps the critical properties of the one-component model onto those of the spherical model. Whether this property does or does not have a deep meaning is an open question, which the authors leave to further investigations.

IV. THE RELATIONSHIP BETWEEN THE EXACT CRITICAL VALUES $-\theta_c$ AND $-r_c$

We have used two different symbols $-r_c$ and $-\theta_c$ to distinguish (respectively) between the critical value obtained by the resummation of certain special diagrams and the exact critical value that would be obtained from the resummation of *all* diagrams. We are now in a position to discuss if, or to what extent, the two quantities coincide. The first remark is that in four dimensions the critical value $-\theta_c = -24u_0$, calculated to first order in u_0 in the framework of the ϵ expansion, exactly coincides with $-r_c$ [Eq. (13a)]. This might look like a coincidence, due to the special property that in $d = 4$, the universal critical quantities of the ϕ^4 model are the same as the Gaussian model. In other words, the case $d = 4$ might suggest that the relationship

$$\theta_c = r_c[1 + O(u_0^a)] \tag{14}$$

work only when the resummation of the diagrams in Fig. 1 yields the *exact* critical exponent. However, it can be shown that Eq. (14) holds true in three dimensions too. We make use of the Ginzburg criterion (revisited above) that the Gaussian behavior is recovered when the relevant field is much larger than u_0^2. In particular, if $s^*_{\text{true}}(\theta)$ is the true (unknown) expression of s^* and $\theta \equiv r_0 + \theta_c$ is the true relevant field [Eq. (2)], then

$$s^*_{\text{true}}(\theta)|_{r_0=0} = s^*_{\text{true}}(\theta_c) = \sqrt{\frac{\theta^+}{\theta_c}}\,[1 + O(u_0^2/\theta_c)], \tag{15}$$

provided $\theta_c \gg u_0^2$. The value $\theta^+ \gg u_0$ corresponds (modulo corrections of order u_0^2) to $s^*_{\text{true}}(\theta^+) = 1$, that is, to the value of the relevant field at which the minimum possible value of the scaling parameter is attained. It is not difficult to verify, by means of Eq. (13b), that the approximated value of s^* at $r_0 = 0$ in three dimensions is

$$s^*(0) = \sqrt{\frac{r_0^+}{36u_0}}\,[1 + O(u_0)], \tag{16}$$

with $s^*(r_0^+) = 1$ (modulo corrections of order u_0^2). On the other hand, we have already claimed that the expression (16) has been obtained from the diagrams in Fig. 1, which yield the largest contribution at $r_0 = 0$. Hence we can take the right-hand side members of Eqs. (15) and (16) to be equal to the lowest significant order in u_0, which yields

$$\theta_c = 36u_0 \frac{\theta^+}{r_0^+}[1 + O(u_0^a)]. \tag{17}$$

Note that the condition $\theta_c \gg u_0^2$ follows self-consistently from Eq. (17). Furthermore, one has, by definition, $\theta^+ = r_0^+ + \theta_c$ and $r_c = 36u_0$ for $d = 3$. Now it is an easy matter to obtain Eq. (14) from Eq. (17) even in three dimensions. This proves that, in three and four dimensions, the exact critical point $-\theta_c$ and the one-loop critical point $-r_c$ coincide to *first* order in u_0.

V. TWO-LOOP APPROXIMATION

In FTR, the two-loop approximation aims to iterate the mass renormalization to second order in u_0. In order to dress the relevant field at the same order, we adopt the same procedure as in the one-loop approximation (Sec. II), that is, we identify the diagrams whose resummation yields

$$f^{(1)}_{\text{res}}(s, r_1) \to f^{(1)}_{\text{res}}(s, r_2), \quad r_2 = r_1 + u_0^2 g(s, r_2) \tag{18}$$

(with g a suitable function), so that the argument of the logarithmic part of Φ_d [Eq. (6)] corresponds to the (renormalized) direct correlation function at the second order in u_0. The diagrams are those reported in Fig. 2 and give the dressed relevant field at two-loop order:

$$r_2 = r_1 + 48\,\frac{d^2 u_0^2}{\Omega_d^2}[r_2\sigma_2(s, r_2) - 2\sigma_0(s, r_2)], \tag{19}$$

where Ω_d is the d-dimensional solid angle and

$$\sigma_0(s, r) \equiv \sigma(x = 0, s, r), \quad \sigma_2(s, r) \equiv \left.\frac{\partial^2 \sigma(x, s, r)}{\partial x^2}\right|_{x=0},$$

$$\sigma(x, s, r) \equiv \int_{\text{out}_s}\int_{\text{out}_s} d^d y\; d^d z \times \frac{\chi_{\text{out}_s}(x + y + z)}{[(x + y + z)^2 + r](y^2 + r)(z^2 + r)}. \tag{20}$$

χ_{out_s} is the characteristic function of the hyperspherical shell:

$$\mathrm{out}_s=\{x\in \mathsf{R}^d; 1/s<|x|\leqslant 1\}.$$

As expected, the dressed field of Eq. (19) coincides with the renormalized mass (see Sec. 6-7 of [3]) [7]. As for the residual free energy at two-loop order, we get

$$f_{\mathrm{res}}^{(2)}(s,r_2)=f_{\mathrm{res}}^{(1)}(s,r_2)+u_0^2\Delta f_2(s,r_2), \tag{21}$$

with Δf_2 a complicated function not reported here for the sake of brevity. The calculation leading to Eqs. (19)–(21) involves a careful account of the anomalous dimension η since some of the diagrams in Fig. 2 also affect the coefficient of x^2 in the integral expressions. Details on these aspects can be found in the E-PAPS file [5], accompanying the present paper.

Let us now discuss the two-loop approximation in the *three-dimensional* case only. For this aim, we do not need to minimize $f_{\mathrm{res}}^{(2)}(s,r_2)$ with respect to s (which is indeed far from trivial) and then study the divergence of the resulting $s^*(r_2)$. The reason is that Eq. (19) itself, in $d=3$, excludes any possible divergence of s^* for $r_2\to 0$. This is due to the fact that the integral σ_0 in Eqs. (19) and (20) behaves, in this limit, as [5]

$$\begin{aligned} &r_2\to 0,\quad \frac{1}{s^*\sqrt{r_2}}<\infty \Rightarrow \sigma_0\sim 4\pi^2\ln 2|\ln\, r_2|^2,\\ &r_2\to 0,\quad \frac{1}{s^*\sqrt{r_2}}\to\infty \Rightarrow \sigma_0\sim 16\pi^2\ln 2|\ln\, r_2|\ln\, s^* \end{aligned} \tag{22}$$

(whereas the term $r_2\sigma_2$ is convergent). In addition, if one believes that the minimization of $f_{\mathrm{res}}^{(2)}(s,r_2)$ yields an inverse power law relation between r_2 and s^* (as in the one-loop case), then the two cases in Eq. (22) are seen to coincide. Therefore, Eq. (19) in the limit $r_2\to 0$ cannot be satisfied by any *finite* r_c. In this sense, the three-dimensional case at two-loop order is quite similar to the two-dimensional case at one-loop order [Eq. (13c)]. In practice, the two-loop approximation in $d=3$ looks *worse* than the one-loop approximation discussed in Secs. II and III. Of course, the same difficulty is encountered in FTR since the equations are formally the same. The only difference is that in FTR the quantity s^* now plays the role of the ''subtraction point'' (denoted as κ in [3]). The basic reason why in $d\leqslant 3$ the situation gets worse, with *increasing* order of approximation, is fairly clear in the present framework: The loop expansion in not an expansion in powers of u_0 alone, but involves the quantity s^* as well. However, we know that s^* *diverges* like the correlation length at the critical point [1] and the way s^* enters the expansion depends crucially on the dimension d. The results obtained so far show that in $d\leqslant 3$, the true ''smallness parameter'' of the expansion becomes divergingly large at the critical point, just at two-loop order ($d=3$) or even at one-loop order ($d=2$). Indeed, these results can be regarded as special cases of the following general argument. The second case in Eq. (22) turns out to be the unique case when s is considered an independent *finite* parameter. Let us take for granted that, at each order, the diagrams containing superficial divergences are all canceled by the dressing procedure itself (or by the so-called *counterterms* in FTR language [3,6]). Then the relevant field is dressed only by the ''most connected'' diagrams. Since these contribute to the two-point direct correlation function with n loops at order n, we are left with a totally coupled $d\times n$-dimensional integral. The n integrated momenta, distributed over $2n-1$ internal lines (denominators), can be cast in a dn-dimensional hypervector X, whose integration domain is roughly $(1/s,1)$ in modulus. The denominators yield a contribution that roughly behaves like $\|X\|^{-2(2n-1)}$ and the (dominant) contribution to the dressed relevant field r_n at the the nth loop should be

$$u_0^n\int_{1/s^*}^1 d\rho\ \rho^{nd-1}\rho^{-4n+2}=u_0^n\int_{1/s^*}^1 d\rho\ \rho^{-n\epsilon+1}, \tag{23}$$

with $\rho=\|X\|$ and $\epsilon=4-d$. From Eq. (23) we see that the quantity

$$\delta_n(s^*)\propto u_0\left(\int_{1/s^*}^1 d\rho\ \rho^{-n\epsilon+1}\right)^{1/n} \tag{24}$$

should describe the true smallness parameter of the loop expansion (see also [3], Sec. 8-4). Indeed, both cases $\epsilon=2$, $n=1$ and $\epsilon=1$, $n=2$ studied above agree with Eq. (24) in that the smallness parameter diverges at the critical point. All this shows that the loop-expansion technique cannot approach the problem close to the critical region, unless one finds a way to keep the divergence of the smallness parameter under control. As stressed in [3] (Sec. 8-4), this necessity preludes to the introduction of the ϵ expansion.

VI. CONCLUSIONS

In [1] a modified version of WRGT was introduced, in which the correlation length is calculated by minimizing the *residual* (renormalized) free energy with respect to the scaling parameter s. As stressed in [1], the method makes use of the basic ingredients of WRGT, but avoids, in principle, the necessity of *iterating* the procedure to approach the *fixed point*. This is because the value of the scaling parameter s^* is determined, with its own critical properties, by the minimization of the residual free energy.

In view of more elaborate applications, a first step is to identify which procedure is to be intended as the *loop expansion* for the present method. The resummation of the diagrams in Fig. 1 is shown to realize the goal at the one-loop level. The present approach yields some progress beyond the standard results. One important point is to recognize that the diagrams in Fig. 1 are dominant for large s, at each order in u_0, if $r_0=0$. It is this result that makes it possible to claim that the one-loop shifting r_c of the critical point [Eq. (1)] is actually the first-order approximation (in u_0) of the exact critical point θ_c (Sec. IV). A revisitation of the Ginzburg criterion (Sec. III) yields the standard results reported in textbooks (see, in particular, [3]). However, an intriguing aspect does emerge, not explicitly stressed by other authors: The one-loop approximation maps the universal critical properties of a *one-component* model onto those of a *spherical* model. This point probably deserves some attention and is left to future investigations.

The strategy used in Sec. II to identify which diagram resummation is equivalent to the one-loop approximation can be extended to next orders: One has to dress the relevant field the same way as the mass is renormalized in FTR. This means that the direct correlation function (two-point vertex function), calculated at higher and higher orders in u_0, must be adsorbed into the *logarithmic* part of the residual free energy. Following this line, it is found that the diagrams to be resummed further at the two-loop level are those in Fig. 2 (Sec. V). In three dimensions, it can be seen that the second-order term in u_0 diverges essentially as a power of $\ln s^*$ near the critical point. This utters the failure of the loop expansion as a useful method to investigate the critical region. Indeed, the expression of the true smallness parameter δ_n of the loop expansion at the nth order [Eq. (24)] shows that δ_n diverges with diverging s^* at any order $n>1$ in $d\leqslant 3$. However, s^* *must* diverge at the critical point since we have shown that it scales like the correlation length. In practice the smallness parameter is never small in $d\leqslant 3$, close enough to the critical point. This clearly explains why in $d\leqslant 3$ the loop expansion is not sufficient, by itself, to improve the approximation order by order if one is interested in the critical region, unless a new perturbative parameter is introduced keeping the divergence of s^* under control. This clarifies the crucial role played by the ϵ expansion.

At this stage one may wonder whether the ϵ expansion is a unique technique to approach the study of the critical properties in $d<4$. The present approach shows that the origin of the problems stems from the *Gaussian* cumulant expansion. The point is that a Gaussian cumulant expansion yields some problems just close to the critical region, where the ''perturbation'' u_0 becomes the *dominant* term. The quickest way to remove any singularity in a perturbative expansion close to the critical point should be treating the Gaussian part itself of the Hamiltonian as a perturbation. Though this approach might look discouraging at first, our next attempt will point to this direction since the *physical* meaning of s^* seems to open some perspectives for a possible *non-Gaussian* (quartic) expansion in cumulants.

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[2] K. G. Wilson and J. Kogut, Phys. Rep. **12**, 75 (1974).

[3] D. J. Amit, *Field Theory, the Renormalization Group, and Critical Phenomena* (World Scientific, Singapore, 1984).

[4] N. D. Mermin and H. Wagner, Phys. Rev. Lett. **17**, 1133 (1966).

[5] See AIP document No. E-PAPS: E-PLEEE8-58-009811 for details and a discussion about the calculations leading to some of the results reported in the main paper. E-PAPS document files may be retrieved free of charge from our FTP server (http://www.aip.org/epaps/epaps.html) or from ftp.aip.org in the directory /epaps/. For further information, e-mail: paps@aip.org or fax: 516-576-2223.

[6] M. Le Bellac, *Des Phénomènes Critiques aux Champs de Jauge* (InterEditions/Editions du CNRS, Paris, 1988).

[7] It should be noticed that the identification of r_2 with the usual renormalized mass is not complete in as much as our approach does not need to regularize the four-point vertex function (quartic coupling constant).